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The factor 1/16 of Bohm diffusion

M I Sanduk
Department of Physics, University of Baghdad, Al Jadriya,
P O. Box 47036, Baghdad, Iraq

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Abstract : Since the semiempirical equation of Bohm diffusion (D_B) does not imply the instability, it cannot explain the same behavior of the turbulent diffusion (D_γ) in inhomogeneous magnetic field.

Attributing Bohm diffusion to plasma turbulence, and considering $D_B = D_\gamma$, we are lead to assume that the factor 1/16 of Bohm's equation is an instability factor. We thus trace the origin of the factor introduced by Bohm.

Keywords : Plasma diffusion, Bohm diffusion, plasma losses

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1. Introduction

As the magnetic field confines the plasma, it may induce a plasma instability. This new state (instability) causes plasma loss, and was first reported by Bohm [1]. The semiempirical formula of Bohm for this anomalous diffusion is

$$D_B\,[\mathrm{m^2/sec}] = T_e\,[\mathrm{eV}]/16\,B\,[\mathrm{T}]. \tag{1}$$

The fraction 1/16 'has no theoretical justification but is an empirical number agreeing with most experiments to within a factor of two or three' [2]. This equation is not formally derived, and there have been many attempts to get the same form of the equation considering plasma instability [3,4].

D_B is independent of density (n) and inversely proportional to B. The value of D_B as Bohm mentioned, is 'intrinsically uncertain within an order of magnitude because of the dependence of fluctuation amplitudes on boundary condition' [1].

Historically, the correctly explained *anomalous* diffusion was that driven by helical instability of the positive column (weakly ionized plasma) by Kadomtsev and Nedospasov in 1960 [5], and the problem was experimentally investigated by Allen *et al.* [6].

Kadomtsev [7] and Dupree [8] showed that the diffusion coefficient according to the quasi-linear theory or by considering wave-particle interaction, is associated with the growth rate (γ) of instability, and has the form

$$D_\gamma = \gamma / 2k_x^2, \tag{2}$$

where $k_x^2 = 2\pi / \lambda_x$ is the wave number of the instability, and D_γ is the turbulent diffusion coefficient. Inspite of Bohm's diffusion as ascribed to the instability, the instability is not implied explicitly in Bohm's form.

Most of the experimental results on diffusion are compared with Bohm's diffusion. For example, Bodin and Newton [9] gave an order of $D_B/10$ or $D_B/100$ for the measured coefficient in the linear Thetatron. In toroidal geometry (San diego dc octupole), Prater *et al* [10] showed a scale of diffusion as

$$D = \frac{B}{B_\theta} \frac{D_B}{500},$$

where B_θ and B are the poloidal and the total magnetic field respectively.

In this attempt, we consider Bohm's diffusion in a gradient magnetic field (∇B) for two reasons : (i) to note the variation of local value of D_B, (ii) since instability condition is $\nabla B * \nabla p < 0$, where p is the plasma pressure, a turbulent diffusion does arise, and D_B can be compared with D_γ.

∇B could be either normal ($\nabla B \perp B$) or parallel ($\nabla B \parallel B$) to B. Since we are interested in the normal diffusion (across B), the perpendicular ∇B has been considered and our aim is to focus on the Bohm's fraction.

2. Diffusion equation in ∇B

Fluid diffusion is attributed to the density gradient (Fick's law). Then Bohm's diffusion equation is given by

$$\nabla * \Gamma = -(\nabla D_B \nabla n + D_B \nabla^2 n), \tag{3A}$$

where Γ, D, and ∇n are the flux of particles, diffusion coefficient and the density gradient respectively. For eq. (1) one obtains

$$\nabla * \Gamma = -1/16 [\nabla n(\nabla T / B - T \nabla B / B^2) + \nabla^2 n\, T / B], \tag{3B}$$

where $\nabla B = \nabla |B|$. This is a general form of Bohm diffusion equation. Then the variation of D_B with the distance is given by

$$\nabla D_B = 1/16 [\nabla T / B - T \nabla B / B^2]. \tag{4}$$

The theoretical explanation of Bohm's diffusion that have been given by Spitzer [3] and Bernstien [4], assumed a fluctuated electric field ($\hat{E}$) that may cause the particles drift ($\hat{E} \times B$). The fluctuation range (λ_x) of $\hat{E}$ is large compared with the Larmor radius (r_l) *i.e.* $\lambda_x > r_l$ and its interval (τ) is long compared to $1/\omega_c$ (where ω_c is cyclotron frequency), $\tau > 1/\omega_c$ [3] The magnetic field is assumed to be in the Y-direction.

D_B may be written in the following form

$$D_B = C T_e / B. \tag{5}$$

According to Spitzer, $C = 2K_1^2 K_2 K_3$, where the Ks are unknown constants of proportionalities. C may be expressed as

$$C = 2\frac{\hat{\phi}}{T}\frac{r_l}{\lambda}\frac{\omega_c}{f} = 2\frac{\hat{\phi}}{T}\frac{v_{Th}}{\hat{v}}, \tag{6}$$

where $f = 1/\tau$ and ϕ is the fluctuated potential.

Accordingly, the concept of Ks may lead us to assume that C is a B-dependent quantity. Then eq. (3B) could be rewritten as

$$\nabla * \Gamma = -[\nabla n(\nabla T C / B + T \nabla C / B - TC \nabla B / B^2) + \nabla^2 n T / B]. \tag{7}$$

Thus, a general form of eq. (4) becomes

$$\nabla D_B = \nabla T C / B + T \nabla C / B - TC \nabla B / B^2. \tag{8}$$

3. What is C ?

At the edge of plasma, an isothermal case is acceptable. Then eq. (8) becomes

$$\nabla D_B = T \nabla C / B - TC \nabla B / B^2. \tag{9}$$

The first term explains the slope due to variation of C. This term modifies the slope of the second one.

We have no clear idea about the behavior of C with the variation of B. Since C is related to the instability, and the total diffusion flux is $\Gamma = \Gamma_0 + \hat{\Gamma}$, where $\hat{\Gamma} = <\hat{n}\, \hat{V}_{\mathrm{EXB}}>$ and the angular bracket indicates averaging with respect to time, we obtain

$$D_B = CT / B \equiv <\hat{n}\, \hat{V}_{EXB}> / \nabla n. \tag{10}$$

For the slab geometry, we assume B in Y-, ∇n in X-, and E to be in Z-direction. One then finds with the aid of eq. (2) that

$$\hat{\Gamma} = (-\gamma / 2\omega^2) \left|k_z \hat{\phi} / B\right|^2 \nabla n_o. \tag{11}$$

Thus,
$$CT = (\gamma / 2\omega^2 B) \left|k_z \hat{\phi}\right|^2. \tag{12}$$

In terms of fluctuation level $(\hat{\phi} / T)$, we get

$$C = \left(\gamma k_z / 2\omega^2\right) \left|k_z \hat{\phi} / B\right| \hat{\phi} / T. \tag{13}$$

This equation shows that the factor C depends on

(i) the fluctuation level of the mode;

(ii) the ratio of growth rate and the frequency.

These features are in agreement with some of those of Spitzer [3].

4. Experimental results

The experimental investigations of anomalous diffusion (turbulent diffusion D_T) for many types of plasma confinement devices [11–14], have been reported.

In most reports, the diffusion coefficients D_B and D_T are obvious functions of distance, that is perpendicular with the magnetic field. Calculated diffusion coefficients as a function of distance are shown in Figure 1 where Figure 1a is for Wisconsin octupole [11]

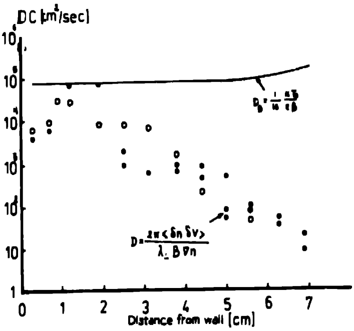

Figure 1(a). Calculated diffusion coefficient as a function of distance, for Wisconsin toroidal octupole [11].

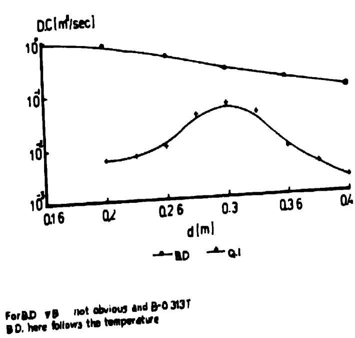

Figure 1(b). Calculated diffusion coefficient as a function of distance, for Culham levitron [12], D_B estimated from results depicted in their Figures 2b and 3a.

and Figure 1b is for Culham levitron [12], and D_B is estimated from the results depicted in their Figures 2b and 3a. Figure 1c is for Ruhr Univ. mirror device [13], and D_B is estimated from given data of the plasma parameters. These results show that :

(i) D_B and D_T do not display the similar variation with the position,

and (ii) they may not have same order of magnitude (in some cases there is an agreement at positions where the instability is high).

UMIST steady-state quadrupole has been described by Phillips *et al.* [15]. The magnetic field is inhomogeneous and fluctuations have been observed in the quadrupole

[16–18]. The region outside the critical surface (plasma edge) has been studied by the author earlier [14] where MHD instability (flute mode of 1 kHz) was observed.

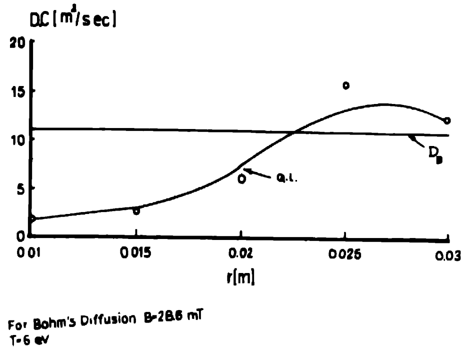

Figure 1(c). Calculated diffusion coefficient as a function of distance, for Ruhr Univ. mirror device [13], D_B estimated from given data of the plasma parameters.

The instability growth rate of the flute mode was estimated by many techniques [14]. Results shown in Figure 2 are the theoretical and the experimental estimations of the growth rate γ.

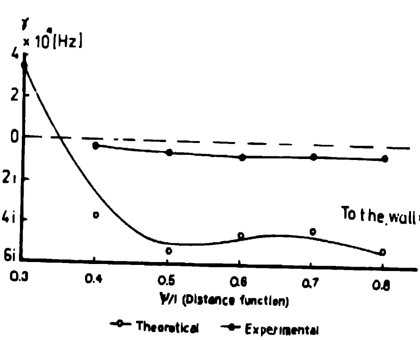

Figure 2. Theoretical growth rate as a function of ψ / l. The correction is for collisions and particle drifts. The dominant mode is flute mode which is about 1kHz. Drift instability was founded to be about 22–35 kHz [16].

The anomalous diffusion coefficients $D(\gamma)$ and $D(\hat{E}\times B)$ were determined. The results showed a good agreement and both exhibit a positive behavior in the instable region (Figure 3).

One expects to find a positive slope of D_γ in the region $\psi/l \simeq$ 0.3–0.6, instead of the negative one of D_B. The maximum D_γ separates the two regions of instability; the regions of growth and the nearly saturation. That is quite clear from the theoretical estimation of the growth rate as shown in Figure 2.

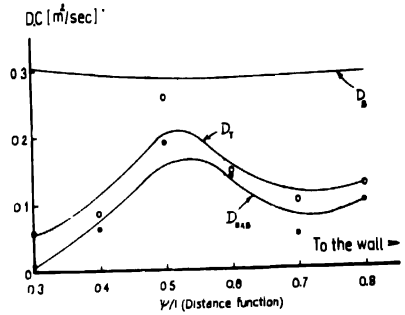

Figure 3. Profile of anomalous diffusion coefficients as a function of ψ/l (flux-function coordinate). Bohm's coefficient D_B compared with two experimental coefficients D ($\hat{E} \times B$) and $D(\gamma)$.

In comparing the behavior of both curves D_B and D_γ, we note that

(i) ∇D_B has an opposite behavior of ∇D_γ. (ii) D_γ follows the instability growth (iii) Both curves have the same region of $\nabla D = 0$, that represents a surface of constant diffusion.

For UMIST quadrupole, eq. (13) becomes :

$$< C > \; = \left(\gamma k_z / 2\omega^2\right) \left| k_z \hat{\phi} U_2 / U_1 \right| \hat{\phi} / T, \tag{14}$$

where $<C>$ is the average along a closed field line, $U_1 = \oint dl / B$ (l the field line length), and $U_2 = \oint dl / B^2$. Figure 4 shows the variation of $<C>$ in the quadrupole radial coordinate (ψ / l) at the region of MHD instability.

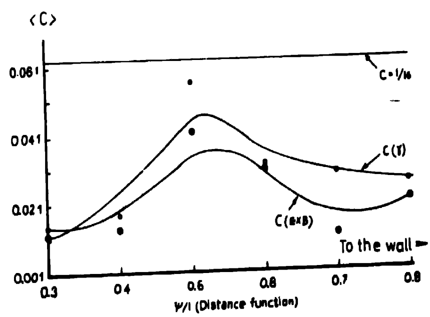

Figure 4. A comparison between Bohm's factor (1/16) and the fluctuation factor $< C >$, where $< C >$ is a function of ψ / l. $< C >$ is estimated from $< \tilde{n} \times \tilde{v} >$ and the growth rate (γ).

By considering the formula (2.13b) of Fischer and Kramer [13], one can find the same form of eq. 13.

4.1. Why is Bohm's factor C = 1/16 ?

In consequence of the above discussion, we are led to suggest that the Bohm's factor (1/16) is due to the fluctuation properties of Bohm's group discharge. In his explanation of the drain diffusion, Bohm used the following parameters ([1], p. 64) : a magnetic field of 0.3T and random electric field (p. 364) of order 3×10^2 V/m, then the drain velocity ($\hat{E} \times B$) becomes 10^3 m/sec. The frequency of the field is 2.5×10^5 Hz, then the wavelength is of the order of 4×10^{-3} m. The electron temperature is 3 eV.

In addition to the above parameters, we need the magnitudes of fluctuation level ($\hat{\phi} / T$) and the growth rate (γ), to apply eq. (17). So we can assume a case of strong turbulence, where $\gamma = \omega$ *i.e.* 10^5 Hz $\leq \gamma \leq 2.5 \times 10^5$ Hz. For the fluctuation level ($\hat{\phi} / T$), it is possible to assume that the measured ratio of the fluctuated current to the dc value (saturation ion current of the probe) is equal to the potential fluctuation level. In their work [1], they had dealt with a level of order 30% (P. 353 & 361).

Accordingly, it is easy to find by using eq. (13) that, $1/20 \leq C \leq 1/8$, for the case mentioned by Bohm [1]. In other hand, calculation based on eq. (10) shows that $C = 6 \times 10^3$, which is far from Bohm's estimation. The first estimation shows that 1/16 is an average value of C. The result agrees with Bohm's words 'the exact value of D_B is *uncertain* within a factor of 2 or 3'. The uncertainty is a feature of the fluctuation, that is nestled in the factor 1/16.

5. Conclusion

Testing D_B using Bohm's formula of the fraction 1/16 in ∇B, shows a contradiction in the behavior with the prediction of quasi-linear theory. The contradiction may be solved by assuming that the factor 1/16 is not just a simple proportional constant. It should be an important dimensionless factor depending on the instability.

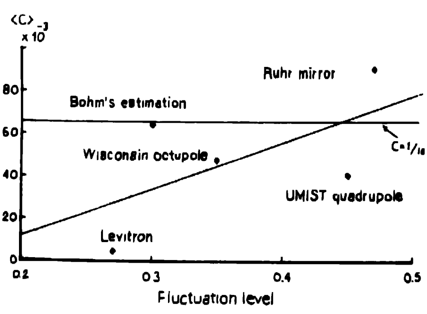

Figure 5. The dependency of the factor C on the fluctuation level of the instability for many measurements made in different plasma systems. $< C >$ is calculated from maximum diffusion coefficient.

Assuming $k_z / \omega = k_z \hat{\phi} / B = \hat{v}$, C of eq. (13) becomes

$$C = \frac{1}{2} \frac{\hat{\phi}}{T} \frac{\gamma}{\omega}. \tag{15}$$

A comparison of eq. (15) with eq. (6) shows that the dependency on the fluctuation level $(\hat{\phi} / T)$ is quite obvious in both Spitzer's model [3] and that of quasi-linear explanation.

Figure 5 depicts the dependency of the factor C (calculated for maximum D_γ as $C = B\ D_\gamma / T$) on the fluctuation level for many measurements made in different plasma systems.

The application of eqs. (6) and (13) on Bohm's data shows a good agreement with the result of eq. (13) with the factor 1/16. Therefore, it is evident that the ratio of growth rate to the frequency has a great influence on C, where this ratio determines the applicability of quasi-linear approximation.

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